Mech 221 Math week 7 suggested problem solutions.

§3.4 #17 $(t+1)^2 y'' - 4(t+1)y' + 6y = 0.$ ★

$y_1(t) = (t+1)^2.$ $\quad y_1' = 2t+2$

a) Look for $y_2 = u y_1 = u(t+1)^2$

$y_2' = (t+1)^2 u' + 2(t+1)u.$

$y_2'' = (t+1)^2 u'' + 4(t+1)u' + 2u.$

plug into ★, algebra.

$(t+1)^4 u'' = 0.$

so $u'' = 0 \Rightarrow u = \underline{\underline{a}} + bt.$

gives a multiple of y_1.

So $y_2(t) = t(t+1)^2$ [could also take $y_2(t) = (t+1)^3$].

b)

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} (t+1)^2 & t(t+1)^2 \\ 2(t+1) & \underbrace{2t(t+1) + (t+1)^2}_{(3t+1)(t+1)} \end{vmatrix}$$

$\dots = (3t+1-2t)(t+1)^3 = (t+1)^4$

$W \neq 0$ except at $t=-1$, expected since

c) $y'' - \frac{4}{t+1} y' + \frac{6}{(t+1)^2} y = 0$

§ 3.6 #4 undamped Spring begins moving from equilibrium position.

$x(t) = A \sin \omega t$

$\frac{dx}{dt} = \omega A \cos \omega t$

$\frac{dx}{dt}(0) = \omega A = 20$ (m/s).

maximum extension $A = 0.2$ (m).

$\Rightarrow \omega = 10$, so $\sqrt{\frac{k}{m}} = 10$

$\Rightarrow k = \underbrace{100}_{s^{-2}} m = \underbrace{100}_{s^{-2}} \cdot \underbrace{20}_{kg} = 2{,}000\, kg/s^2$.

#12 $my'' + \gamma y' + ky = 0$.

a) $mr^2 + \gamma r + k = 0$,

roots $r = \dfrac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m}$

as given, overdamped, r_1 and r_2 real < 0.

$y(t) = Ae^{r_1 t} + Be^{r_2 t}$

$y(0) = y_0 \Rightarrow A + B = y_0$.

$y'(t) = r_1 A e^{r_1 t} + r_2 B e^{r_2 t}$

$y'(0) = 0 \Rightarrow r_1 A + r_2 B = 0$.

$A = \dfrac{r_2 y_0}{r_2 - r_1}$ $\qquad B = \dfrac{r_1 y_0}{r_1 - r_2}$.

b) write $\sqrt{\gamma^2 - 4km}$

$= \gamma \sqrt{1 - \underbrace{4km/\gamma^2}} \approx \gamma\left(1 - \dfrac{2km}{\gamma^2}\right)$

$= \gamma - \dfrac{2km}{\gamma}$.

γ large so
this is small, use
$\sqrt{1-\varepsilon} \approx 1 - \frac{\varepsilon}{2}$
(linear, Taylor)

use in expression at top of page

So $r \approx \dfrac{-\gamma \pm (\gamma - 2km/\gamma)}{2m}$.

$$r_1 \approx -\frac{k}{\gamma} \qquad \lim_{\gamma \to \infty} r_1 = 0.$$

$$r_2 \approx -\frac{\gamma}{m} \qquad \lim_{\gamma \to \infty} r_2 = -\infty.$$

so as $\gamma \to \infty$ the expression for $y(t)$ becomes

$$y(t) \approx A e^{r_1 t} \approx y_0.$$

c) Physically, the system becomes so damped, the spring can't pull the mass back to equilibrium.

§ 3.10 #3. a) $k \Delta x = mg$

$$k = \frac{(10)(9.8)}{0.098} = 1000. \text{ kg}/s^2.$$

$\uparrow$ convert to m

b) $m y'' + ky = 20e^{-t}.$

$$y'' + 100y = 2e^{-t}.$$

$y_p = ae^{-t}$ $\quad y_p'' = ae^{-t}$ $\quad$ plug in to get

$$101 a = 2 \Rightarrow a = \frac{2}{101}.$$

$$y = \underbrace{A\cos 10t + B\sin 10t}_{y_c} + \underbrace{\frac{2}{101}e^{-t}}_{y_p}.$$

$$y(0) = A + \frac{2}{101} = 0 \Rightarrow A = -\frac{2}{101}$$

$$y'(t) = -10A\sin 10t + 10B\cos 10t - \frac{2}{101}e^{-t}$$

$$y'(0) = 0 \Rightarrow B = \frac{1}{10}\cdot\frac{2}{101}$$

So $y = \frac{2}{101}\left(-\cos 10t + \frac{1}{10}\sin 10t + e^{-t}\right)$.

c) Solution is bounded as $t \to \infty$. To find the max ($|y| \approx 0.035$ m) either graph the solution, or check $y(t)$ values where t solves $y'(t) = 0$ [find by N's method].

#5 a) $k = 1000$ kg/s^2 as above.

b) $my'' + ky = 20$ $\quad (0 \le t \le \pi)$.

$$y_p = \frac{20}{k} = 0.02$$

$y = 0.02(1 - \cos 10t)$. $\quad y' = 0.2\sin 10t$

at $t = \pi$, $y' = 0$, $y = 0$, so $y \equiv 0$ for $t > \pi$.

c) $|y_{max}| = 0.04.$

\# 15 $\quad V_s = L \dfrac{dI}{dt} + \dfrac{1}{C} Q \qquad \left(\dfrac{dQ}{dt} = I\right).$

$L = 1, \quad \dfrac{1}{C} = \dfrac{1}{4}$

$V_s = 5 \sin 3t$

differentiate,

$$I'' + \frac{1}{4} I = V_s' = 15 \cos 3t.$$

$$\left.\begin{aligned} I_p &= a \cos 3t + b \sin 3t \\ I_p'' &= -9a \cos 3t - 9b \sin 3t \end{aligned}\right\}$$

plug into the equation.

$$\left(-9 + \frac{1}{4}\right) a = 15, \qquad b = 0.$$

$$a = -\frac{12}{7}.$$

$$I = A \cos\left(\frac{t}{2}\right) + B \sin\left(\frac{t}{2}\right) - \frac{12}{7} \cos 3t.$$

$$I = 0, \quad I' = 0 \Rightarrow A = \frac{12}{7}, \quad B = 0.$$

$$I = \frac{12}{7}\left(\cos(t/2) - \cos(3t)\right).$$

#17. $V'' + \frac{1}{RC} V' + \frac{1}{LC} V = \frac{1}{C} \frac{dI_s}{dt}$

$= \frac{1}{C} e^{-t}.$

$V'' + 2V' + 2V = 2e^{-t}$

$r^2 + 2r + 2 = 0 \Rightarrow r = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i.$

$V_c = e^{-t}(A \cos t + B \sin t).$

$V_p = a e^{-t} \qquad V_p' = -a e^{-t} \qquad V_p'' = a e^{-t}$

$a(1 - 2 + 2) = 2 \Rightarrow a = 2.$

$V = e^{-t}(A \cos t + B \sin t) + 2 e^{-t}$

$V(0) = 0 \Rightarrow A + 2 = 0 \Rightarrow A = -2.$

$V' = -e^{-t}(A \cos t + B \sin t)$

$+ e^{-t}(-A \sin t + B \cos t) - 2e^{-t}$

$V'(0) = 0 \Rightarrow -A + B - 2 = 0 \Rightarrow B = 0,$

so $V = 2e^{-t}(1 - \cos t).$