Mech 221 Math Suggested Problems Solutions
week #5, Brian Wetton.

§ 7.4 #3. $y' = ty^2$ $\quad y(0) = 1.$ $\quad h = \frac{1}{10}.$

a) 3RK $\quad K_1 = 0$

$$K_2 = \frac{h}{2}(1+0)^2 = \frac{1}{20}$$

$$K_3 = h(1 - 0 + 2h K_2)$$

$$= \frac{1}{10}\left(1 + \frac{1}{100}\right) = 0.101$$

$$y(h) \approx 1 + \frac{h}{6}(K_1 + 4K_2 + K_3) \approx 1.0050167$$

b) 4RK $\quad K_1 = 0$

$$K_2 = \frac{h}{2}(1+0)^2 = \frac{1}{20}. = 0.05$$

$$K_3 = \frac{h}{2}\left(1 + \frac{h}{2}\left(\frac{1}{20}\right)\right) = 0.050125$$

$$K_4 = h(1 + hK_3) = 0.10050125$$

$$y(h) \approx 1 + \frac{h}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$$\approx 1.005013.$$

c) Neither 3RK nor 4RK should be exact since $y^{(4)}(t) \neq 0$ and $y^{(5)}(t) \neq 0$.

d) $y_{exact}(0.1) = \dfrac{2}{2-0.01} = \dfrac{2}{1.99}$

$y_e^{(0.1)} - y_{3RK}(0.1) \approx 8.4\times10^{-6}$

$y_e^{(0.1)} - y_{4RK}(0.1) \approx 1.2\times10^{-5}$

The 3RK result is more accurate. This is not expected until one notices that the solution is an even function of t. Thus, $f^{(4)}(0)=0$, and so $f^{(4)}$ will be small near $t=0$. Since the errors from 3RK are proportional to the size of $f^{(4)}$, the 3RK errors are small in this particular case.

#15 $\quad y''' = ty \qquad y(0)=1,\ y'(0)=0,\ y''(0) = -1.$

$h=0.1$

rewrite as a first order system for

$y_1 = y,\ \ y_2 = y',\ \ y_3 = y''$: $\qquad$ Like in Math 152.

$y_1' = y_2,\ \ y_2' = y_3,\ \ y_3' = t y_1.$

or in vector form

$$\underline{y}' = \begin{bmatrix} y_2 \\ y_3 \\ t y_1 \end{bmatrix} \leftarrow \underline{f}(t, \underline{y}).$$

$$\underline{y}_0 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}. \qquad \underline{K}_1 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}.$$

$$\underline{K}_2 = \underline{f}\left(\frac{h}{2}, \underline{y}^0 + \frac{h}{2}\underline{K}_1\right) = \begin{bmatrix} -0.05 \\ -1.0 \\ 0.05 \end{bmatrix}$$

$$\underline{K}_3 = \underline{f}\left(\frac{h}{2}, \underline{y}^0 + \frac{h}{2}\underline{K}_2\right) = \begin{bmatrix} -0.05 \\ -0.9975 \\ 0.049875 \end{bmatrix}.$$

$$\underline{K}_4 = f\left(h, \underline{y}_0 + h\underline{K}_3\right) = \begin{bmatrix} -0.09975 \\ -0.9950125 \\ 0.0995 \end{bmatrix}.$$

$$y(0.1) \approx \underline{y}_1 = \underline{y}_0 + \frac{h}{6}\left(\underline{K}_1 + 2\underline{K}_2 + 2\underline{K}_3 + \underline{K}_4\right)$$

$$\approx \begin{bmatrix} 0.995004 \\ -0.099834 \\ -0.995013 \end{bmatrix}$$

#19 $y' = \frac{t}{y+1}$ $y(0) = 1$.

Exact $y(t) = -1 + \sqrt{t^2+4}$

$y(1) = \sqrt{5} - 1$.

4RK 20 steps $h = \frac{1}{20}$.
(in matlab).

$y_{20} \approx 1.2360679$

Error $y(1) - y_{20} \approx 8 \times 10^{-8}$

#26 Reformulate as a system $y_1 = y$, $y_2 = y'$,
so

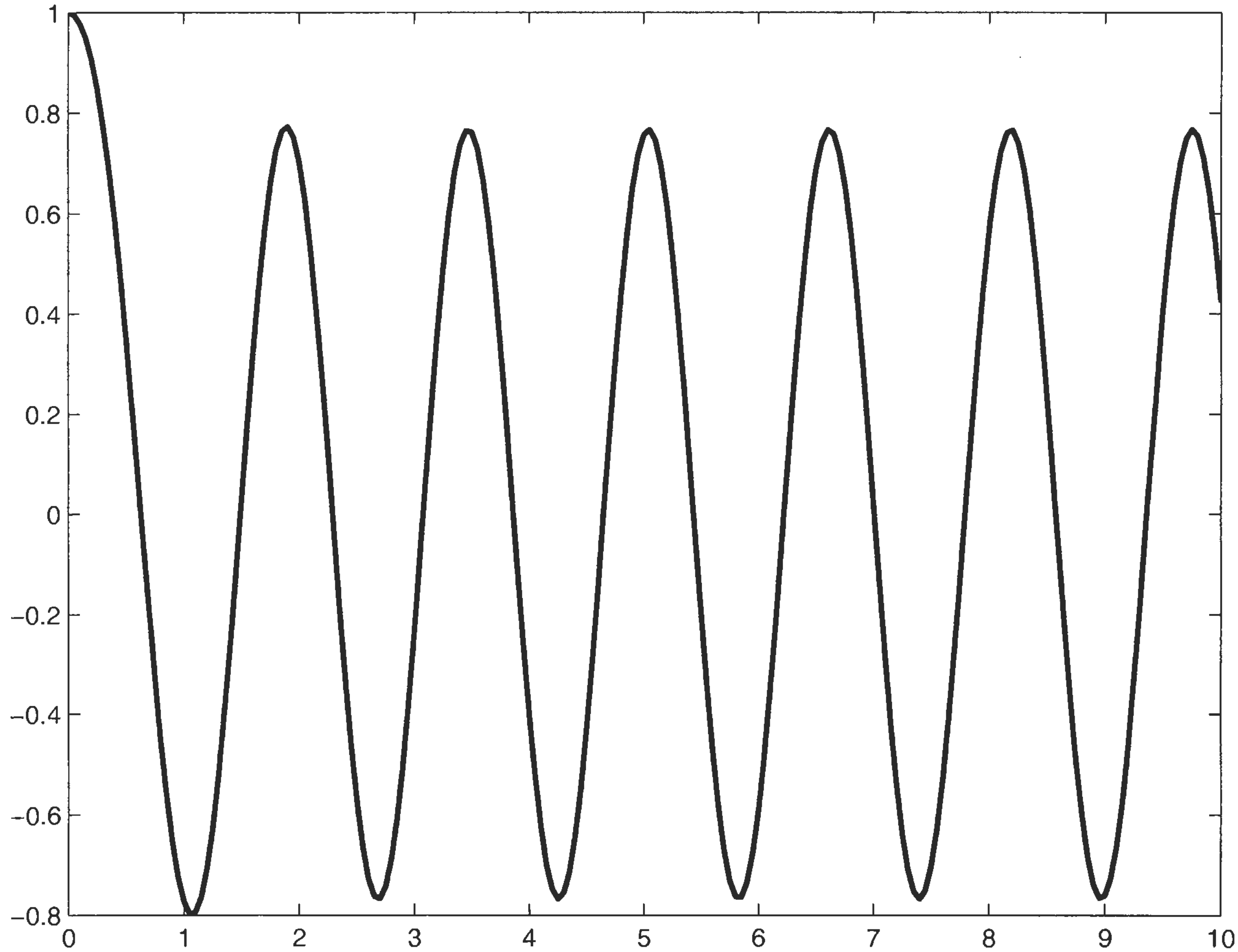
$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} y_2 \\ -4(1+3\tanh t)\, y_1 \end{pmatrix}$$

$$\underline{y}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Soln in MATLAB, soln does tend to period $\pi/2$ oscillations as suggested by the limit of the equation to

$$y'' + 16y = 0 \qquad \text{as } t \to \infty.$$

Graph next page.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