Mech 221 Math week #3 Homework Solutions.

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§2.2 #13 $y' - 2\cos(2t)y = 0 \quad y(\pi) = -2.$

use formula to get general solution,

$p(t) = -2\cos(2t) \qquad P(t) = \int p(t)dt = -\sin(2t).$

$y(t) = C e^{\sin(2t)}$

#15 $\dfrac{y'}{(t^2+1)y} = 3 \qquad y(1) = 4.$

rewrite $y' - 3(t^2+1)y = 0.$

$p(t) = -3t^2 - 3 \qquad P(t) = -t^3 - 3t.$

$y(t) = C e^{t^3+3t}$

#7 $2y' + 3y = e^t$ ⊛ $y(0) = 0$.

$y' + \frac{3}{2}y = \frac{1}{2}e^t$

constant coefficient, can use MUC. ← using formula OK too.

$y_0(t) = Ce^{-3/2t}$

$\left.\begin{aligned} y_p(t) &= ae^t \\ y_p'(t) &= ae^t \end{aligned}\right\}$ plug in ⊛ to get $5a = 1$, $a = \frac{1}{5}$.

So $y(t) = y_0 + y_p = \frac{1}{5}e^t + Ce^{-3/2t}$

$y(0) = 0 \Rightarrow C = -\frac{1}{5}$ so $y(t) = \frac{1}{5}(e^t - e^{-3/2t})$.

#8 $y' + y = 1 + 2e^{-t}\cos 2t$ (+) $y(\pi/2) = 0$.

Constant coefficient, can use MUC here also, but using the formula gives much easier algebra. Multiply (+) by integrating factor e^t.

$\frac{d}{dt}(e^t y) = e^t + 2\cos 2t$ ↓ integrate.

$e^t y = e^t + \sin 2t + C$ ← $y(\pi/2) = 0 \Rightarrow C = -e^{\pi/2}$

$y = 1 + e^{-t}\sin 2t - e^{-t+\pi/2}$.

#9 $2y' + (\cos t)y = -3\cos t \qquad y(0) = -4.$

Could use formula directly, but let me use a trick.

$$2y' + \cos t\,(y+3) = 0 \qquad \text{Let } v = y+3.$$

$$v' + \tfrac{1}{2}\cos t\, v = 0.$$

change makes problem homog.

$$v(t) = C e^{-\frac{1}{2}\sin t}$$

$$y(t) = -3 + C e^{-1/2 \sin t} \qquad y(0) = -4$$

$$y(t) = -3 - e^{-1/2 \sin t} \qquad \Rightarrow C = -1.$$

#40 $y' + p(t)y = 2 \quad y(0) = 1 \quad p(t) = \begin{cases} 0 & 0 \le t \le \\ \frac{1}{t} & 1 \le t \le 2 \end{cases}$

For $0 \le t \le 1$,

$$y' = 2,\ y(0) = 1 \Rightarrow y(t) = 1 + 2t.$$

$$y(1) = 3.$$

IC for next part.

$1 \le t \le 2 \qquad y' + \frac{1}{t}y = 2. \qquad$ IF t.

$$(ty)' = 2t \Rightarrow ty = t^2 + C$$

$$y = t + C/t \qquad y(1) = 3 \Rightarrow C = 2$$

so $y(t) = t + 2/t.$

§2.6 #3 $\quad y' + \frac{1}{y+1} = 0 \qquad y(1) = 0.$

separable $\quad y' = -\frac{1}{y+1}$

$(y+1)\,dy = -dx$ $\quad$ integrate

$\frac{y^2}{2} + y = -x + C \qquad y(1) = 0 \Rightarrow C = 1.$

$y^2 + 2y + (2x-2) = 0$

$y(x) = -1 \pm \sqrt{1-(2x-2)} = -1 + \sqrt{3-2x}$

$\uparrow$

take + solution to match IC.

#9 $\quad \frac{dy}{dt} = t - ty^2 = t(1-y^2) \qquad y(0) = \frac{1}{2}.$

$\frac{dy}{1-y^2} = t\,dt$ $\quad$ partial fractions $\quad \frac{1}{1-y^2} = \frac{-1/2}{y-1} + \frac{\frac{1}{2}}{y+1}.$

integrate

$\frac{1}{2}\ln\left|\frac{y+1}{y-1}\right| = \frac{t^2}{2} + C \qquad y(0) = \frac{1}{2} \Rightarrow C = \frac{1}{2}\ln 3.$

Solving for y... $\quad \ln\left|\frac{y+1}{y-1}\right| = t^2 + \ln 3.$ $\quad$ careful, $|\cdot|$

$\frac{y+1}{y-1} = 3e^{t^2} \Rightarrow y+1 = -(y-1)3e^{t^2}.$ $\quad$ negative so $y(1+3e^{t^2}) = 3e^{t^2}-1.$

$$y(t) = \frac{3e^{t^2}-1}{3e^{t^2}+1}.$$

#30 a) $y' = y^3$

b) $y' = y(4-y)$

c) $y' = -y^2.$

#2 a) $mv' = -mg + Kv^2$ (v negative, resistance opposes acceleration downwards)

$V' = -g + \frac{K}{m}V^2$ separable

$$\frac{dV}{-g+\frac{K}{m}V^2} = dt \quad (A)$$

partial fractions

$$\frac{1}{\frac{K}{m}V^2 - g} = \frac{M}{K}\,\frac{1}{V^2 - \frac{gm}{K}} = \frac{M}{K}\left(\frac{1}{V-\sqrt{\frac{gm}{K}}} - \frac{1}{V+\sqrt{\frac{gm}{K}}}\right)$$

times $\frac{1}{2}\sqrt{\frac{K}{gm}}$.

Use this form to integrate (A).

$$\frac{1}{2}\sqrt{\frac{m}{Kg}}\ \ln\left|\frac{V-\sqrt{gm/k}}{V+\sqrt{gm/k}}\right| = t.$$

Note: $-\sqrt{gm/k}$ is the terminal velocity.
Again note that $|\cdot|$ is negative above.

$$\frac{\sqrt{gm/k}-V}{\sqrt{gm/k}+V} = e^{2\sqrt{kg/m}\,t} + c \quad \leftarrow \quad V(0)=0 \Rightarrow c=0.$$

algebra... (similar to last question)

$$V(t) = -\sqrt{\frac{gm}{k}}\ \frac{e^{2\sqrt{kg/m}\,t}-1}{e^{2\sqrt{kg/m}\,t}+1}.$$

Note: There are other forms of the answer.
Check that $V(0)=0$, and $\lim_{t\to\infty} V(t) = -\sqrt{\frac{gm}{k}}$.

b). As above, terminal velocity is $\sqrt{gm/k}$, so

$$\sqrt{\frac{200\,lb}{k}} = 10\,mi/h = 14.67\ ft/sec \quad \text{(units - ugh!)}$$

$$\text{So } K \approx 0.929\ \frac{lb\ sec^2}{ft^2}$$