Trigonometry

\[
\sin \theta = \frac{o}{h} \\
\cos \theta = \frac{a}{h} \\
\tan \theta = \frac{o}{a} \\
\tan x = \frac{\sin x}{\cos x} \\
\sin^2 x + \cos^2 x = 1 \\
\sin(-x) = -\sin x \\
\cos(-x) = \cos x \\
\sin(x + y) = \sin x \cos y + \cos x \sin y \\
\cos(x + y) = \cos x \cos y - \sin x \sin y \\
e^{ix} = \cos x + i \sin x
\]

Quadratic Equation

\[ax^2 + bx + c = 0\]

has roots

\[x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\]

Newton’s Method

If an initial guess \(x_0\) is close enough to a root of a function \(g\), then the iteration formula

\[x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}\]

gives increasingly good estimates \(x_n\) of the root.
Elementary Derivatives

\[ \frac{d}{dx} x^r = r x^{r-1} \quad (r \neq 0) \]
\[ \frac{d}{dx} \sin x = \cos x \]
\[ \frac{d}{dx} \cos x = -\sin x \]
\[ \frac{d}{dx} \tan x = \sec^2 x = \frac{1}{\cos^2 x} \]
\[ \frac{d}{dx} e^x = e^x \]
\[ \frac{d}{dx} \ln |x| = \frac{1}{x} \]
\[ \frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}} \]
\[ \frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1-x^2}} \]
\[ \frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2} \]

Taylor Polynomials and Series

Taylor polynomial approximation:

\[ f(x) \approx P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \]

Residual formula

\[ f(x) - P_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1} \]

where \( \xi \) is a point between \( a \) and \( x \) (that is not known).

Basic Taylor (McLaurin) series:

\[
\begin{align*}
\sin x &= x - \frac{x^3}{6} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\
\cos x &= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \\
e^x &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{4!} + \cdots \\
\frac{1}{1-x} &= 1 + x + x^2 + x^3 + \cdots \quad \text{for } |x| < 1
\end{align*}
\]

Numerical Integration

Approximations to

\[ I = \int_a^b f(x)dx \]

starting from a division of \([a, b]\) into \( N \) sub-intervals of equal length \( h = (b-a)/N \):

Trapezoidal Rule:

\[ I \approx T_N = \frac{h}{2} f(a) + hf(a+h) + hf(a+2h) + \cdots + hf(b-h) + \frac{h}{2} f(b) \]
with error expression

\[ I - T_N = -\frac{f''(\xi)}{12}(b-a)h^2 \]

**Simpson’s Rule:**  \((N\) must be even\)

\[
I \approx S_N = \frac{h}{3} f(a) + \frac{4h}{3} f(a+h) + \frac{2h}{3} f(a+2h) + \frac{4h}{3} f(a+3h) + \frac{2h}{3} f(a+4h) + \cdots + \frac{4h}{3} f(b-h) + \frac{h}{3} f(b)
\]

with error expression

\[ I - S_N = -\frac{f^{(4)}(\xi)}{180}(b-a)h^4 \]

In the error expressions above, \(\xi\) is a point between \(a\) and \(b\) (that is not known).

**Numerical Differentiation**

**Forward:**  \(f'(a) \approx (f(a+h) - f(a))/h\).

**Backward:**  \(f'(a) \approx (f(a) - f(a-h))/h\).

**Centred:**  \(f'(a) \approx (f(a+h) - f(a-h))/(2h)\).

**Centred second derivative:**  \(f''(a) \approx (f(a+h) - 2f(a) + f(a-h))/h^2\).

**Approximating Differential Equations**

To find approximations \(y_k\) to \(y(kh)\) where \(y(t)\) solves

\[ \frac{dy}{dt} = f(y,t) \]

and \(h\) is a small fixed step, the following methods can be used:

**Forward Euler:**  \(y_{k+1} = y_k + hf(y_k, kh)\)

**Backward Euler:**  \(y_{k+1} = y_k + hf(y_{k+1}, (k+1)h)\)

**Linear Interpolation**

If \(f(a)\) and \(f(b)\) are known and \(c\) is in \([a, b]\) then

\[ f(c) \approx \frac{b-c}{b-a} f(a) + \frac{c-a}{b-a} f(b) \]
Eigen-Analysis

Eigenvalues of a matrix $A$ are scalar values $\lambda$ that solve

$$\det(A - \lambda I) = 0.$$ 

If $\lambda$ is an eigenvalue then $x$ is called a corresponding eigenvector if $x$ is nonzero and solves

$$(A - \lambda I)x = 0$$

(that is, $Ax = \lambda x$).

Differential Equations

Scalar, linear, first order

$$y' = a(t)y + f(t)$$

with initial data $y(0) = y_0$. Let

$$A(t) = \int_0^t a(\tau)d\tau.$$ 

The solution is given by

$$y(t) = e^{A(t)y_0} + e^{A(t)} \int_0^t e^{-A(\tau)} f(\tau)d\tau.$$

Scalar, linear, second-order constant coefficient

$$ay'' + by' + cy = f(t)$$

Solution is $y(t) = y_0(t) + y_p(t)$ (homogeneous plus particular).

homogeneous: Solve auxiliary equation:

$$ar^2 + br + c = 0.$$ 

Three cases:

1. Two distinct real roots $r_1$ and $r_2$:

$$y_0(t) = Ae^{r_1t} + Be^{r_2t}.$$ 

2. Repeated real root $r$:

$$y_0(t) = Ae^{rt} + Bte^{rt}.$$ 

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3. Complex conjugate roots \( r = a \pm ib \):

\[
y_o(t) = e^{at}(A \cos bt + B \sin bt).
\]

**particular:** If \( f(t) \) has one of the following forms, the Method of Undetermined Coefficients can be used:

- **If \( f(t) \) is a polynomial in \( t \) of order \( n \):** take \( y_p(t) \) to also be a polynomial in \( t \) of order \( n \).
- **\( f(t) = \sin \omega t \) or \( f(t) = \cos \omega t \):** take
  \[
y_p(t) = a \sin \omega t + b \cos \omega t.
\]
- **\( f(t) = e^{bt} \):** take
  \[
y_p(t) = ae^{bt}.
\]

**special case (resonance):** If any one of the terms in the form for the particular solution above is in the homogeneous solution, multiply the form of \( y_p(t) \) above by \( t \) until this is no longer true.

**solving for the coefficients:** Insert the form of \( y_p(t) \) into the differential equation to solve for the undetermined coefficients in \( y_p(t) \). Then (and only then) find \( A \) and \( B \) from the complete solution \( y = y_o + y_p \) using the initial (or boundary) data.

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**Vector, first order, linear, homogeneous, constant coefficient (diagonalizable)**

\[
y' = Ay
\]

with initial data \( y(0) = y_o \). Assume that \( A \) is a diagonalizable matrix,

\[
A = EDE^{-1}
\]

where \( D \) is the diagonal matrix of eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_n \) and \( E \) is the matrix with the corresponding eigenvectors in columns. Then the solution is

\[
y(t) = EME^{-1}y_o
\]

where \( M \) is the diagonal matrix with entries \( e^{\lambda_1 t}, e^{\lambda_2 t}, \ldots, e^{\lambda_n t} \).
Special case: Defective Matrix

In the case above when \( A \) is a diagonalizable matrix, the solution \( y \) is written as a linear combination of terms

\[ x_i e^{\lambda_i t} \]

where \( \lambda_i \) are the eigenvalues of \( A \) and \( x_i \) the corresponding eigenvectors. It can happen for some matrices \( A \) (known as defective matrices) that for certain eigenvalues \( \lambda \) that are repeated twice, there is only one eigenvector \( x \). In this case, find the vector \( z \) (a generalized eigenvector) that satisfies

\[ (A - \lambda I)z = x. \]

The solution of

\[ y' = Ay \]

contains linear combinations of

\[ xe^{\lambda t} \quad \text{and} \quad (tx + z)e^{\lambda t} \]

in this case.