Mech 221 Mathematics Component

Differential Equations

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Lectures 24-25

Outline

Lecture 24

- Method of Undetermined Coefficients
- Examples

Lecture 25

- Variation of Parameters
- Examples
- Formula for Second Order Scalar Problems

Outline

Method of Undetermined Coefficients

Linear, Constant Coefficient, Homogeneous Systems

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}(t)$$

Remember, the general solution of the inhomogeneous problem can be written

$$\mathbf{x}(t) = \mathbf{x}_p(t) + \mathbf{x}_c(t) = \mathbf{x}_p(t) + \mathbf{\Phi}(t)\mathbf{c}$$

where $\mathbf{\Phi}(t)$ is a fundamental solution. There are two methods to find particular solutions:

1. For specific kinds of functions $\mathbf{f}(t)$ there is the Method of Undertermined Coefficients.
2. Variation of parameters.

Method of Undetermined Coefficients-II

- The Method of Undetermined Coefficients for linear first order systems is the same as that for first and second order scalar problems *except* that the coefficients in $\mathbf{f}$ are vectors and the unknown parameters in $\mathbf{x}_p$ are also vectors.
- The resonant case is tricky to explain, we'll see what happens in examples.

Examples

Example 1

Find the general solution to

$$\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Examples-II

Example 1 (cont.)

Examples-III

Example 2

Find the general solution to

$$\mathbf{x}' = \begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2\sin t \\ -e^{-3t} \end{bmatrix}$$

Examples-IV

Example 2 (cont.)

Examples-V

Example 2 (cont.)

Examples-VI

Example 3

Find the general solution to

$$\mathbf{x}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2e^{-t} \\ -e^{-t} \end{bmatrix}$$

Examples-VII

Example 3 (cont.)

Examples-VIII

Example 3 (cont.)

Outline

Variation of Parameters

Formula

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t) \text{ with } \mathbf{x}(t_0) = \mathbf{x}_0$$

If $\mathbf{\Psi}(\mathbf{t})$ is a fundemental solution (to the homogeneous problem) then the solution to the IVP above is

$$\mathbf{x}(t) = \mathbf{\Phi}(t) \int_{t_0}^{t} \mathbf{\Phi}^{-1}(s)\mathbf{f}(s)ds + \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(t_0)\mathbf{x}_0$$

You can recognize this as being in the form:

$$\mathbf{x}(t) = \mathbf{x}_p(t) + \mathbf{x}_c(t)$$

Note that this formula does not require **A** to be constant coefficient.

Variation of Parameters-II

Derivation

To derive the Variation of Parameters formula, begin by considering

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{u}(t)$$

with $\mathbf{u}(t)$ to be determined.

Variation of Parameters-III

Derivation (cont.)

Examples

Example 1

Find the general solution of

$$\mathbf{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix}$$

Examples-II

Example 1 (cont.)

Examples-III

Example 1 (cont.)

Examples-IV

Example 2

Find the general solution of

$$\mathbf{x}' = \begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2\sin t \\ -e^{-3t} \end{bmatrix}$$

Examples-V

Example 2 (cont.)

Examples-VI

Example 2 (cont.)

Formula for Second Order Scalar Problems

Consider the second order problem

$$\ddot{x} + p(t)\dot{x} + q(t)x = f(t).$$

Two independent solutions $x_1(t)$ and $x_2(t)$ to the *homogeneous* problem are known. Write as a first order system and use Variation of Parameters to find a formula for a particular solution to this problem.

Formula for Second Order Scalar Problems-II

Formula for Second Order Scalar Problems-III