Mech 221 Mathematics Component

Differential Equations

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Outline

Outline

Systems of First Order Linear Equations

Form of Equations

Here, there are are n unknowns

$$x_1(t), x_2(t), \dots x_n(t)$$

that satify equations

$$\begin{aligned}
x_1' &= a_{11}(t)x_1(t) + a_{12}(t)x_2(t) + \dots + a_{1n}x_n(t) + f_1(t) \\
x_2' &= a_{21}(t)x_1(t) + a_{22}(t)x_2(t) + \dots + a_{2n}x_n(t) + f_2(t) \\
&\vdots \\
x_n' &= a_{n1}(t)x_1(t) + a_{n2}(t)x_2(t) + \dots + a_{nn}x_n(t) + f_n(t)
\end{aligned}$$

where the functions $a_{ij}(t)$ and $f_j(t)$ are given.

Systems of First Order Linear Equations-II

Where do These Come From?

- Coupled systems of oscillators.
- LRC networks.
- Coupled chemical reaction and mixing processes.
- Ecological models, population growth, epidemics.
- Discretizations of continuum models.
- Higher order equations and systems can be reduced to first order systems.

The size n of systems in practical applications can be *very* large.

Systems of First Order Linear Equations-III

Example

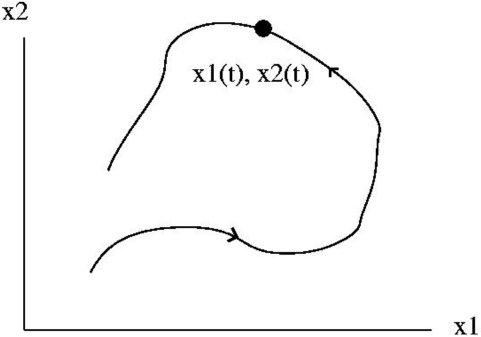
Convert the linear equation

$$\frac{d^4u}{dt^4} - u = 0$$

to a first order linear system.

Phase Plane

In two dimensions, we can plot a solution as a parametric curve $(x_1(t), x_2(t))$.



In this diagram the $x_1 - x_2$ plane is called the *phase* plane.

Phase Plane-II

Discussion

- Information is lost in the phase plane picture - the times at which the solution is at each point on the curve.
- This can be compensated somewhat by labelling some points with the times they correspond to, or at least adding arrows with the direction of increasing time.
- The phase plane is particularly useful in the *autonomous* case, where none of the a_{ij} or f_j depend on t. In this case, trajectories in the phase plane can't cross.
- Phase plane analysis is also useful for autonomous *nonlinear* systems with two unknowns.

Phase Plane-III

Example

Consider the equation (underdamped spring)

$$\ddot{x} + \dot{x} + x = 0$$

1. Find the solution when $x(0) = 1$ and $\dot{x} = 0$.
2. Write the equation as a first order system.
3. Plot the solution you found in part #1 in the phase plane.

Phase Plane-IV

Example (cont.)

Phase Plane-V

Example (cont.)

Matrix Form of the Equations

The system was labelled so it can naturally be written

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}$$

A solution is a vector $\mathbf{x}(\mathbf{t})$.

Matrix Form of the Equations

Example

Write the linear system corresponding to

$$\frac{d^4u}{dt^4} - u = 0$$

in matrix form.

Matrix Form of the Equations

Superposition

Linear systems obey the usual superposition principle. If $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ solve the problems

$$\begin{aligned}\mathbf{x}_1(t) &= \mathbf{A}(t)\mathbf{x}_1 + \mathbf{f}_1(t)\\ \mathbf{x}_2(t) &= \mathbf{A}(t)\mathbf{x}_2 + \mathbf{f}_2(t)\end{aligned}$$

(same matrix, different forcing terms) then $\mathbf{w} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$ solves

$$\mathbf{w}(t) = \mathbf{A}(t)\mathbf{w} + c_1\mathbf{f}_1(t) + c_2\mathbf{f}_2(t)$$

(same matrix, linear combination of forcing terms).

In particular, this shows that linear combinations of solutions to the *homogeneous* problem ($\mathbf{f} \equiv 0$) are also solutions of the homogeneous problem.

Matrix Form of the Equations

Matrix of Homogeneous Solutions

If $\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_n$ are all (homogeneous) solutions of

$$\mathbf{x}' = \mathbf{A}\mathbf{x}$$

then the principle of superposition shows that

$$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \ldots + c_n\mathbf{x}_n$$

is also a homogeneous solution. This can be written in the compact form

$$\mathbf{x} = \boldsymbol{\Phi}\mathbf{c}$$

where $\mathbf{c}$ is the column vector of c values and $\boldsymbol{\Phi}$ is the matrix with the vectors $\mathbf{x}_j$ as columns.

Matrix Form of the Equations

Matrix of Homogeneous Solutions (cont.)

$$\mathbf{\Phi} = [\mathbf{x}_1 | \mathbf{x}_2 | \dots | \mathbf{x}_n]$$

Note that $\mathbf{\Phi}$ solves the vector system:

$$\mathbf{\Phi}' = \mathbf{A}\mathbf{\Phi}$$

(verify column by column).

Outline

Theory of First Order Linear Equations

IVP

The problem for $\mathbf{x}(t)$

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t)$$

where $\mathbf{A}(t)$, $\mathbf{f}(t)$ are given and intial data

$$\mathbf{x}(t_0) = \mathbf{x}_0$$

are also given, has a unique solution in an interval in t around t_0 extending to values of t (if any) where the functions in $\mathbf{A}(t)$ or $\mathbf{f}(t)$ have singularities.

Theory of First Order Linear Equations-II

Complementary (Homogeneous) + Particular

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t)$$

Using the usual superposition argument, the general solution of this problem can be written as

$$\mathbf{x}(t) = \mathbf{x}_c(t) + \mathbf{x}_p(t)$$

where $\mathbf{x}_p(t)$ is any *particular* solution of the inhomogeneous problem (with the $\mathbf{f}$) and $\mathbf{x}_c(t)$ is any *complementary* solution of the homogeneous problem (with $\mathbf{f} \equiv 0$).

Theory of First Order Linear Equations-III

Solving the IVP

$$\mathbf{x}(t) = \mathbf{x}_c(t) + \mathbf{x}_p(t)$$

To solve the IVP, we need to find $\mathbf{x}_c(t)$ such that

$$\mathbf{x}_c(t_0) = \mathbf{x}_0 - \mathbf{x}_p(t_0)$$

Thus, we need to find complementary solutions capable of matching any initial conditions.

The Fundamental Matrix

Let $\mathbf{e}_j$ be the standard basis vectors of $\mathcal{R}^n$:

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \quad \ldots \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Let $\mathbf{x}_j$ be the solution of

$$\mathbf{x}_j' = \mathbf{A}\mathbf{x}_j$$

with $\mathbf{x}_j(t_0) = \mathbf{e}_j$.

The Fundamental Matrix-II

We can make a matrix of these solutions like last lecture

$$\boldsymbol{\Psi} = [\mathbf{x}_1 | \mathbf{x}_2 | \dots | \mathbf{x}_n]$$

Note that $\boldsymbol{\Psi}(t_0) = \mathbf{I}$ (the $n \times n$ identify matrix). $\boldsymbol{\Psi}$ is called *the* Fundamental Matrix. We showed last lecture that for any constant vector $\mathbf{c}$

$$\mathbf{x} = \boldsymbol{\Psi}\mathbf{c}$$

is a solution to the homogeneous problem

$$\mathbf{x}' = \mathbf{A}\mathbf{x}.$$

Fundamental Matrix-III

Solving the IVP

If we take

$$\mathbf{x} = \mathbf{\Psi}\mathbf{x}_0$$

then

$$\mathbf{x}(t_0) = \mathbf{I}\mathbf{x}_0 = \mathbf{x}_0$$

so we have a nice formula for the solution of the initial value problem for any initial conditions.

Fundamental Matrix-IV

Other Fundamental Matrices

Suppose we had n solutions of the homogeneous problem $\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_n$ that we find *somehow*. Put them in a matrix

$$\mathbf{\Phi} = [\mathbf{x}_1 | \mathbf{x}_2 | \ldots | \mathbf{x}_n]$$

as before and consider linear combinations

$$\mathbf{\Phi c}$$

Under what conditions on $\mathbf{\Phi}$ can we solve any IVP?

Fundamental Matrix-V

Other Fundamental Matrices (cont.)

Wronskian

Using $\boldsymbol{\Psi}$ to solve the IVP at another time

Consider again *the* Fundamental Matrix $\boldsymbol{\Psi}$ with $\boldsymbol{\Psi}(t_0) = \mathbf{I}$. Linear combinations are

$$\boldsymbol{\Psi}\mathbf{c}$$

Can we choose $\mathbf{c}$ to solve the IVP given at another time t?

$$\boldsymbol{\Psi}(t)\mathbf{c} = \mathbf{x}_0$$

can be solved for any initial data $\mathbf{x}_0$ as long as $\boldsymbol{\Psi}(t)$ is invertible (equivalent to its determinant is not zero).

Wronskian-II

Abel's Formula

The Wronskian $W(t)$ is the determinant of $\mathbf{\Psi}(t)$. We know $W(t_0) = 1$, we want to show that $W(t)$ is not zero at other times.
Key Lemma (Abel's Formula):

$$W' = W \operatorname{trace}[\mathbf{A}(t)]$$

The trace of a matrix is the sum of its diagonal entries.

Wronskian-III

Abel's Formula (cont.)

Wronskian-IV

Summary

- To solve any IVP for the homogeneous problem we need to find n different (linearly independent) solutions.
- If the solutions are linearly independent at any time, they are linearly independent at all times.
- If the solutions are linearly dependent at any time, they are linearly dependent at all times. In fact, one must be a fixed linear combination of the others.

Outline

Homogeneous Constant Coefficient Systems

Exponential Solutions

Systems like

$$\mathbf{x}' = \mathbf{A}\mathbf{x}$$

where $\mathbf{A}$ is a *constant* $n \times n$ matrix. Experience with scalar linear equations suggests that solutions that are exponential in time may exist. Look for solutions in the form:

$$\mathbf{x}(t) = \mathbf{k}e^{rt}$$

where $\mathbf{k}$ is a constant vector.

Homogeneous Constant Coefficient Systems-II

Exponential Solutions (cont.)

Homogeneous Constant Coefficient Systems-III

Exponential Solutions (cont.)

$$\mathbf{x}' = \mathbf{A}\mathbf{x} \text{ has solutions of the form } \mathbf{x}(t) = \mathbf{k}e^{rt}$$

when r is an eigenvalue of $\mathbf{A}$ and $\mathbf{k}$ is a corresponding eigenvector.
Procedure for Eigenanalysis: First, find r that solve

$$\det(\mathbf{A} - r\mathbf{I}) = 0 \quad \text{roots of an n'th order polynomial}$$

If r is an eigenvalue then a corresponding eigenvector $\mathbf{k}$ is nonzero and solves

$$(\mathbf{A} - r\mathbf{I})\mathbf{k} = 0.$$

Homogeneous Constant Coefficient Systems-IV

Exponential Solutions (cont.)

Recall:

- Eigenvalues of **A** can be real or complex, distict or repeated.
- Eigenvectors from different eigenvalues are linearly independent.
- There is always at least one eigenvector for every eigenvalue. For repeated eigenvalues, you can have the same number of linearly independent eigenvalues as the multiplicity of the eigenvalue (or possibly fewer).
- We will consider in this course only the "nice" case in which **A** has n linearly independent eigenvectors (**A** is diagonalizable).
- In this case, we have n linearly independent solutions to the homogeneous problem. From the theory last lecture, we can construct a Fundamental Solution.

Systems with Real Eigenvalues

Case of n Distinct, Real Eigenvalues

$$\{r_1, r_2, \ldots r_n\}$$

- For each eigenvalue we can find an eigenvector.
- $\mathbf{x}_j = \mathbf{k}_j e^{r_j t}$ solves the homogeneous problem. These vectors are linearly independent at $t = 0$ so

$$\boldsymbol{\Psi} = \left[\mathbf{k}_1 e^{r_1 t} \middle| \mathbf{k}_2 e^{r_2 t} \middle| \cdots \middle| \mathbf{k}_n e^{r_n t}\right]$$

  is a Fundamental solution.
- The general solution of the homogeneous problem is

$$\boldsymbol{\Psi}\mathbf{c} = c_1 \mathbf{k}_1 e^{r_1 t} + c_2 \mathbf{k}_2 e^{r_2 t} + \cdots + c_n \mathbf{k}_n e^{r_n t}$$

Systems with Real Eigenvalues-II

Example 1

Find the general solution of:

$$\begin{aligned}\frac{dx}{dt} &= -3x - 2y \\ \frac{dy}{dt} &= -2x - 6y\end{aligned}$$

and sketch the behaviour of solutions in the phase plane $x - y$.

Systems with Real Eigenvalues-III

Example 1 (cont.)

Systems with Real Eigenvalues-IV

Example 2

1. Find the general solution of:

$$\begin{aligned}\frac{dx}{dt} &= -2y\\ \frac{dy}{dt} &= -2x - 3y\end{aligned}$$

   and sketch the behaviour of solutions in the phase plane $x - y$.

2. Find the solution that satisfies $x(0) = 1, y(0) = \beta$.
3. For what value of β does your solution satisfy $\mathbf{x}(t) \to \mathbf{0}$ as $t \to \infty$?

Systems with Real Eigenvalues-V

Example 2 (cont.)

Systems with Real Eigenvalues-V

Example 3

Find *the* Fundamental Matrix $\boldsymbol{\Psi}(\mathbf{t})$ (for initial conditions at $t = 0$) solution to the previous example. Also write the general formula for $\boldsymbol{\Psi}$ for constant coefficient problems with distinct, real eigenvalues.

Systems with Real Eigenvalues-V

Example 3 (cont.)

Systems with Real Eigenvalues

Matrix Exponential

$\mathbf{\Psi}(t) = \mathbf{T}\mathbf{\Lambda}(\mathbf{t})\mathbf{T}^{-1}$ has the property that

$$\mathbf{x}(t) = \mathbf{\Psi}(t)\mathbf{x}_0$$

solves $\mathbf{x}' = \mathbf{A}\mathbf{x}$ with $\mathbf{x}(0) = \mathbf{x}_0$. Consider the scalar problem

$$x' = ax \quad \text{with } x(0) = x_0$$

with solution $x(t) = e^{at}x_0$. This should tempt us to denote

$$\mathbf{\Psi}(t) = e^{\mathbf{A}t}$$

This is not just formal notation. In fact, it can be shown that

$$\mathbf{\Psi}(t) = \sum_{j=0}^{\infty} \frac{(\mathbf{A}t)^j}{j!}$$

Phase Plane

Stability and Instability of $\mathbf{x} = \mathbf{0}$

Note that $\mathbf{x} \equiv \mathbf{0}$ is an equilibrium solution of

$$\mathbf{x}' = \mathbf{A}\mathbf{x}$$

(the only equilibrium solution unless $\mathbf{A}$ is singular). We are interested in the stability of this equilibrium:

- stable: (asymptotically stable) All solutions approach $\mathbf{0}$ as $t \to \infty$.
- unstable: There are points arbitrarily close to $\mathbf{0}$ such that solutions $\mathbf{x}(t)$ starting with initial values at these points satisfy $|\mathbf{x}(t)| \to \infty$ as $t \to \infty$.

Let's look at what can happen in the $n = 2$ case, where we can graph solutions in the phase plane.

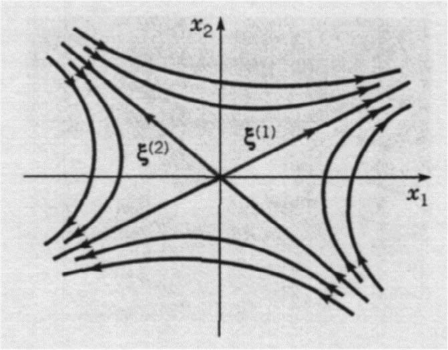
Phase Plane-II

Saddle Point

Real distinct eigenvalues of different signs. The general solution is of the form:

$$\mathbf{x}(t) = c_1\mathbf{k}_1 e^{r_1 t} + c_2\mathbf{k}_2 e^{r_2 t}$$

with $r_1 > 0$ and $r_2 < 0$. In this case, $\mathbf{0}$ is *unstable*.



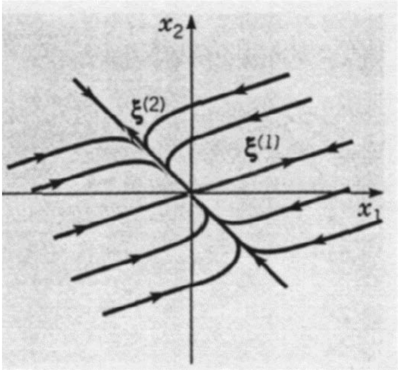
Phase Plane-III

Nodal Point

Real distinct eigenvalues of the signs. The general solution is of the form:

$$\mathbf{x}(t) = c_1\mathbf{k}_1 e^{r_1 t} + c_2\mathbf{k}_2 e^{r_2 t}$$

In this case, **0** is *unstable* if the eigenvalues are positive (nodal source) and stable if the eigenvalues are negative (nodal sink).



Outline

Systems with Complex Eigenvalues

Conjugate Pairs

Complex eigenvalues of real matrices **A** occur in conjugate pairs $r_{1,2} = a \pm ib$. If the eigenvector that corresponds to r_1 is $\mathbf{k} = \mathbf{k}_R + i\mathbf{k}_I$ then it can be shown that the eigenvector corresponding to r_2 is its conjugate $\mathbf{k}_R - i\mathbf{k}_I$.

Systems with Complex Eigenvalues-II

Complex Eigensolutions

Systems with Complex Eigenvalues

Real Eigensolutions

By taking linear combinations of the complex solutions we can find the following independent real homogeneous solutions:

$$\begin{aligned}\mathbf{x}_1(t) &= e^{at}\left(\mathbf{k}_R \cos bt - \mathbf{k}_I \sin bt\right)\\ \mathbf{x}_2(t) &= e^{at}\left(\mathbf{k}_I \cos bt + \mathbf{k}_R \sin bt\right)\end{aligned}$$

Examples

Example 1

Find the general solution of

$$\begin{aligned} \frac{dx}{dt} &= -x + 2y \\ \frac{dy}{dt} &= -2x - y \end{aligned}$$

and sketch the solutions in the phase plane.

Examples-II

Example 1 (cont.)

Examples-III

Example 2

Describe the different possible behaviours of solutions to

$$\frac{dx}{dt} = \alpha x + 2y$$
$$\frac{dy}{dt} = -2x + \alpha y$$

in the phase plane for different values of the parameter α.

Examples-IV

Example 2 (cont.)

Examples-V

Example 3

Find the general solution of

$$\begin{aligned}\frac{dx}{dt} &= x\\ \frac{dy}{dt} &= 2x + y - 2z\\ \frac{dz}{dt} &= 3x + 2y + z\end{aligned}$$

Examples-VI

Example 3 (cont.)

Examples-VII

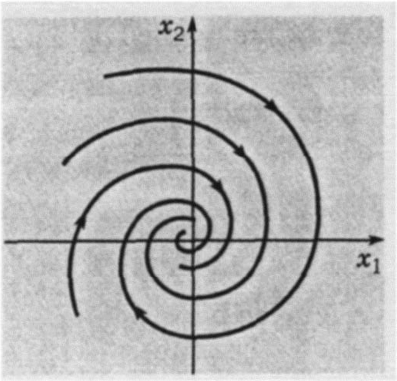
Example 3 (cont.)

Phase Plane

Spiral Point

$$\mathbf{x}(t) = c_1 e^{at}\left(\mathbf{k}_R \cos bt - \mathbf{k}_I \sin bt\right) + c_2 e^{at}\left(\mathbf{k}_I \cos bt + \mathbf{k}_R \sin bt\right)$$

If $a > 0$ the origin is unstable (spiral source). If $a < 0$ the origin is stable (spiral sink).



Phase Plane-II

Centre

If the eigenvalues are purely imaginary then all solutions are periodic. The origin is stable (but not asymptotically stable).