Mech 221 Mathematics Component

Differential Equations

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Lectures 18-19

Outline

Lecture 18

- Spring Mass Systems
- Undamped Case
- Forcing Close to Resonant Frequency

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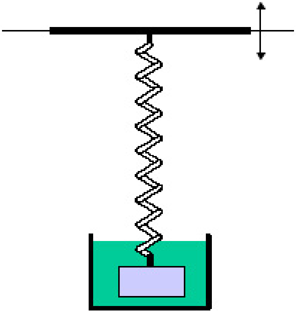
- Spring Oscillations with Damping
- Forced Spring Oscillations with Damping
- Scaling the Damped Spring Equation

Outline

Lecture 18

Spring Mass Systems

- We consider only spring mass systems as shown
- Many other electrical/mechanical vibrations are also governed by the same second order differentials considered here.



Spring Mass Systems-II

Preliminaries

- Let L be the natural length of the spring, m the mass of the weight and $f(t)$ an applied force (upwards).
- Let the length of the spring be denoted by $l(t)$.
- Neglect the mass of the spring itself.
- From Newton's second law:

$$m\frac{d^2 l}{dt^2} = mg + f_d(t) + f_s(t) - f(t)$$

where g is the gravitational acceleration, $f_d(t)$ is the force due to damping and $f_s(t)$ is the force that the spring exterts.

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Spring Mass Systems-III

Preliminaries (cont.)

$$m\frac{d^2 l}{dt^2} = mg + f_d(t) + f_s(t) - f(t)$$

- Hooke's law gives

$$f_s(t) = -k\left[l(t) - L\right]$$

where k is the *spring constant*.

- The damping force $f_d(t)$ is due to friction with the fluid and opposes the motion of the weight. It is assumed that friction is proportional to the speed of the weight:

$$f_d(t) = -\beta\frac{dl}{dt}$$

where β is the damping coefficient.

$$m\frac{d^2 l}{dt^2} = mg - \beta\frac{dl}{dt} - k\left[l(t) - L\right] - f(t)$$

Spring Mass Systems-IV

Preliminaries (cont.)

$$m\frac{d^2 l}{dt^2} = mg - \beta\frac{dl}{dt} - k\left[l(t) - L\right] - f(t)$$

- Consider the equilibrium position $l \equiv L_*$, with no external forcing and no movement of the spring $l' \equiv 0$ and $l'' \equiv 0$:

$$0 = mg - k\left[L_* - L\right]$$

or $L_* = L + mg/k$. L_* is the *equilibrium position* of the spring, mg/k is the *equilibrium extension* of the spring.

- Define $x(t)$ (the distance above the equilibrium position) by

$$l(t) = L_* - x(t)$$

- We obtain

$$m\frac{d^2 x}{dt^2} + \beta\frac{dx}{dt} + kx = f(t)$$

Undamped Case

Simple Harmonic Motion

If there is no damping ($\beta = 0$) and no forcing ($f \equiv 0$) then

$$m\frac{d^2x}{dt^2} + kx = 0$$

The general solution is

$$x(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t.$$

where $\omega_0 = \sqrt{\frac{k}{m}}$ is the natural (angular) frequency of the spring mass system. The period of oscillation is

$$T = \frac{2\pi}{\omega_0}$$

Undamped Case-II

Amplitude-Phase Form of the Solution

$$x(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t.$$

This solution can be written in the form

$$x(t) = R \cos(\omega_0 t - \phi)$$

where R is the *amplitude* and ϕ is the *phase angle*. By convention (but different convetions in different fields) we will take $-\pi < \phi \leq \pi$.

Undamped Case-III

Amplitude-Phase Form of the Solution (cont.)

Undamped Case-IV

Example

The natural length of a spring is 1m. A mass is attached to the end and the length increases to 1.02m at equilibrium. The mass is then displaced upward by 1cm from the equilibrium position and released with upward velocity of 14cm/s.

1. Find the natural frequency and period of the system.
2. Find the displacement $x(t)$.
3. Find the amplitude and phase angle of the motion.

Undamped Case-V

Example (cont.)

Undamped Case-VI

Forced Oscillations

Solve the IVP

$$\ddot{x} + \omega_0^2 x = \cos \omega t \quad \text{with } x(0) = 0 \text{ and } \dot{x}(0) = 0$$

with $\omega \neq \omega_0$

Undamped Case-VII

Resonant Forced Oscillations

Solve the IVP

$$\ddot{x} + \omega_0^2 x = \cos \omega_0 t \quad \text{with } x(0) = 0 \text{ and } \dot{x}(0) = 0$$

Forcing Close to Resonant Frequency

Beats

It seems odd that the solutions of the resonant case are unbounded in t but the non-resonant case are bounded, no matter how close to resonance we are. Let's consider the non-resonant solution more closely:

$$x(t) = \frac{\cos \omega t - \cos \omega_0 t}{\omega_0^2 - \omega^2}$$

It can be shown that this solution can be rewritten in the form:

$$x(t) = \frac{2}{\omega_0^2 - \omega^2} \sin \left[\frac{\omega_0 - \omega}{2} t \right] \sin \left[\frac{\omega_0 + \omega}{2} t \right]$$

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Forcing Close to Resonant Frequency-II

Beats(cont.)

Lecture 18

Forcing Close to Resonant Frequency-III

Beats(cont.)

Forcing Close to Resonant Frequency-IV

Beats(cont.)

$$x(t) = \frac{2}{\omega_0^2 - \omega^2} R(t) \sin\left[\frac{\omega_0 + \omega}{2} t\right]$$

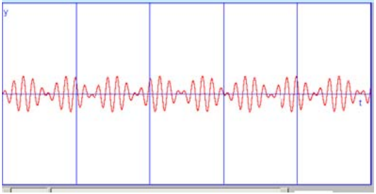
where $R(t) = \sin\left[\frac{\omega_0 - \omega}{2} t\right]$ can be thought of as an amplitude.

- For ω far from ω_0 there is no insight gained by this form.
- For ω close to ω_0 the frequency of $\sin\left[\frac{\omega_0+\omega}{2} t\right]$ approaches the natural frequency. $R(t)$ becomes large and the period $T = 4\pi/|\omega_0 - \omega|$ becomes very large. $|\omega_0 - \omega|/2$ is the beat frequency.

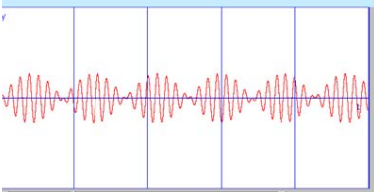
Forcing Close to Resonant Frequency-V

Pictures of Beats $\omega_0 = 54$

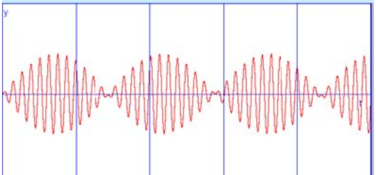
$\omega = 44$



$\omega = 47$



$\omega = 50$



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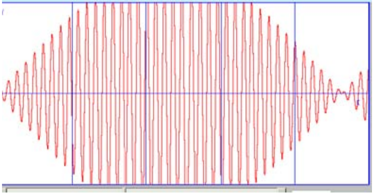
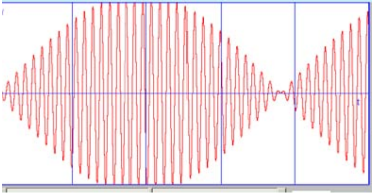
Forcing Close to Resonant Frequency-VI

Pictures of Beats $\omega_0 = 54$ (cont.)

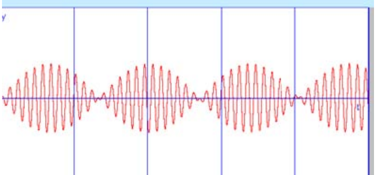
$\omega = 53$



$\omega = 56$

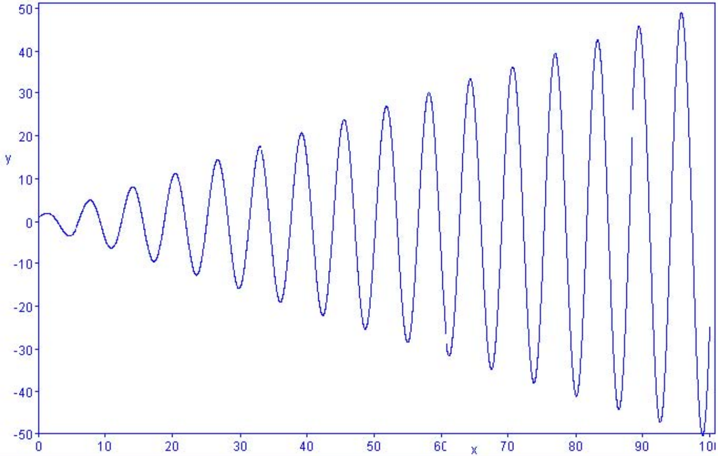


$\omega = 59$



Forcing Close to Resonant Frequency-VII

Picture of Resonance



Outline

Spring Oscillations with Damping

Free Oscillations

$$m\ddot{x} + \beta\dot{x} + kx = 0$$

we consider $\beta > 0$ (positively damped).

1. What happens for “small” and “large” β?
2. What do we mean by “small” and “large” β?

Characteristic equation is:

$$mr^2 + \beta r + k = 0$$

with roots

$$r_{1,2} = \frac{-\beta \pm \sqrt{\beta^2 - 4km}}{2m}$$

Spring Oscillations with Damping-II

Free Oscillations (cont.)

$$r_{1,2} = \frac{-\beta \pm \sqrt{\beta^2 - 4km}}{2m}$$

Three cases:

1. $\beta^2 < 4km$ (small β) roots are complex - *underdamped* motion.
2. $\beta^2 > 4km$ (large β) roots are real - *overdamped* motion.
3. $\beta^2 = 4km$ repeated real roots *critically damped* motion.

Note that for $\beta > 0$ the real parts of $r_{1,2}$ are always both negative. Thus, we always have exponential decay of the solutions as $t \to \infty$.

Spring Oscillations with Damping-III

Overdamped and Critically Damped Cases

Overdamped

$$r_{1,2} = \frac{-\beta \pm \sqrt{\beta^2 - 4km}}{2m}$$

are both real and negative.

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Critically Damped

$$r = -\frac{\beta}{2m}$$

$$x(t) = c_1 e^{rt} + c_2 t e^{rt}$$

Spring Oscillations with Damping-IV

Underdamped

$$x(t) = e^{-\beta t/(2m)} \left[c_1 \cos \omega_1 t + c_2 \sin \omega_1 t \right]$$

where

$$\omega_1 = \sqrt{\frac{k}{m} - \left(\frac{\beta}{2m} \right)^2}$$

is the quasi-frequency. This can also be written in phase-amplitude form:

Spring Oscillations with Damping-V

Example

A mass of 64g stretches a spring 6cm in equilibrium and a dashpot provides a linear damping force to the system.

1. Determine the spring constant k
2. Determine the value of the damping constant β for which the system is critically damped.
3. Find the displacement from equilibrium $x(t)$ if the mass is released from position $x(0) = 3$cm with zero velocity.

Spring Oscillations with Damping-VI

Example (cont.)

Forced Spring Oscillations with Damping

$$m\ddot{x} + \beta\dot{x} + kx = F_0 \cos \omega t$$

The general solution is of the form

$$x(t) = x_c(t) + x_p(t)$$

where $x_p(t)$ is of the form (MUC)

$$x_p(t) = a \cos \omega t + b \sin \omega t$$

and $x_c(t)$ has one of the forms from the previous section (depending on the size of β) all of which decay to 0 as $t \to \infty$. Thus we call $x_c(t)$ the *transient* part of the solution and $x_p(t)$ the *steady state* part.

Forced Spring Oscillations with Damping-II

Example

Assuming m, k, β, and F_0 are constant, what value of the forcing frequency ω produces the largest amplitude steady state response? The smallest steady state response?

Forced Spring Oscillations with Damping-III

Example (cont.)

Forced Spring Oscillations with Damping-IV

Example (cont.)

Forced Spring Oscillations with Damping-V

Example (cont.)

What does the previous example tell you about the speed you should drive on a bumpy road to make the ride as smooth as possible?

Scaling the Damped Spring Equation

$$m\ddot{x} + \beta\dot{x} + kx = F_0 \cos \omega t$$

Scale x and t to make this equation dimensionless and as simple as possible.

Scaling the Damped Spring Equation-II