Mech 221 Mathematics Component

Differential Equations

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Lectures 14-17

Outline

Outline

Second Order DE Applications

General Linear Second Order DE

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = g(t)$$

where $p(t)$, $q(t)$, and $g(t)$ are given functions.

In particular, we will look (again) at a simpler class of linear equations where the coefficients are constants:

$$A\frac{d^2y}{dt^2} + B\frac{dy}{dt} + Cy = g(t)$$

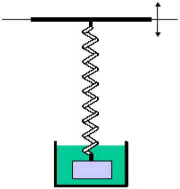
$(A \neq 0)$. Even these simpler DEs occur in many applications.

Second Order DE Applications-II

Applications

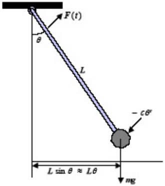
Damped spring systems:

$$m\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = f(t)$$



Small oscillations of a pendulum:

$$mL^2\frac{d^2\theta}{dt^2} + cL\frac{d\theta}{dt} + mgL\theta = F(t)$$

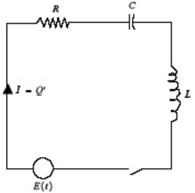


Second Order DE Applications-III

Applications (cont.)

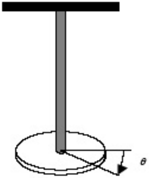
LRC series circuits:

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E(t)$$



Tortional motion of a weight on a twisted shaft:

$$I\frac{d^2\theta}{dt^2} + c\frac{d\theta}{dt} + k\theta = T(t)$$

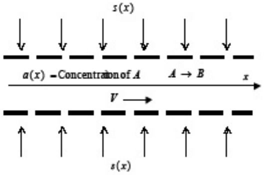


Second Order DE Applications-IV

Applications (cont.)

Combined diffusion, convection and reaction of a chemical in a permeable channel.

$$D\frac{d^2a}{dx^2} - V\frac{da}{dx} - ka - s(x) = 0$$



General Discussion

IVP

The second order linear problem

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = g(t)$$

must also have initial data a and b specified:

$$\begin{aligned} y(t_0) &= a \\ y'(t_0) &= b \end{aligned}$$

The DE and initial data together make an Initial Value Problem (IVP).

General Discussion-II

IVP (cont.)

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = g(t)$$

with $y(t_0)$ and $y'(t_0)$ given.

Q: Why two pieces of data?

1. To solve the problem for y you have to "integrate" twice.
2. If y were a position then the DE gives the acceleration. To specify position given acceleration you need to know the initial position and velocity.

Theory: The IVP above has a unique solution defined in an interval around t_0 up to values of t (if any) where p, q or g have singularities.

General Discussion-III

Superposition

If y_1 and y_2 solve the linear problems below (same DE, different RHS):

$$\begin{aligned}\frac{d^2y_1}{dt^2}+p(t)\frac{dy_1}{dt}+q(t)y_1 &= g_1(t)\\ \frac{d^2y_2}{dt^2}+p(t)\frac{dy_2}{dt}+q(t)y_2 &= g_2(t)\end{aligned}$$

Then the linear combination $w=c_1y_1+c_2y_2$ solves

$$\frac{d^2w}{dt^2}+p(t)\frac{w}{dt}+q(t)w=c_1g_1(t)+c_2g_2(t)$$

(same DE, linear combination of the RHS).

In particular, linear combinations of complementary (homogeneous) solutions (with zero RHS) are also complementary solutions. We could say that complementary solutions form a linear subspace of functions.

General Discussion-IV

Complementary and Particular Solutions

If y_1 and y_2 solve the same linear problem (same DE, same RHS):

$$\begin{aligned}\frac{d^2y_1}{dt^2}+p(t)\frac{dy_1}{dt}+q(t)y_1 &= g(t)\\ \frac{d^2y_2}{dt^2}+p(t)\frac{dy_2}{dt}+q(t)y_2 &= g(t)\end{aligned}$$

Then their difference $w = y_1 - y_2$ solves

$$\frac{d^2w}{dt^2}+p(t)\frac{w}{dt}+q(t)w=0$$

This is the complementary (homogeneous) problem. This shows that any solution y of

$$\frac{d^2y}{dt^2}+p(t)\frac{dy}{dt}+q(t)y=g(t)$$

can be written as

$$y = y_c + y_p$$

Some Theory

Fundamental Solutions

Consider the homogeneous equation:

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0$$

Let

- y_1 be a solution with $y_1(0) = 1$, $y_1'(0) = 0$.
- y_2 be a solution with $y_1(0) = 1$, $y_1'(0) = 0$.

Now, $y = ay_1 + by_2$ solves the homogeneous problem above and satisfies the initial conditions:

$$\begin{aligned} y(0) &= a \\ y'(0) &= b \end{aligned}$$

We have shown that the space of homogeneous solutions is two dimensional.

Some Theory-II

Wronskian

Q: Can the fundamental solutions solve the homogeneous IVP specified at other times?

$$\begin{aligned} y(t_0) &= a \\ y'(t_0) &= b \end{aligned}$$

Let's see. Try $c_1 y_1 + c_2 y_2$ in these initial conditions.

Some Theory-III

Wronskian (cont.)

$$W(t) = y_1 y_2' - y_2 y_1'$$

We know that $W(0) = 1$. We want to show that $W \neq 0$ for other t.

Some Theory-IV

Wronskian (cont.)

Some Theory-V

Wronskian (cont.)

In fact, if $W = 0$ at any value of t then y_1 and y_2 must be the same function (possibly multiplied by a constant).

Some Theory-VI

Wronskian (cont.)

Some Theory-VII

Summary

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0$$

- Look for two different solutions y_1 and y_2.
- The general solution is $c_1 y_1 + c_2 y_2$.
- Any initial conditions at any value of t can be matched by this general solution.

Outline

Homogeneous Linear Constant Coefficient Equations

Exponential Solutions

$$A\frac{d^2y}{dt^2} + B\frac{dy}{dt} + Cy = 0$$

$(A \neq 0)$. From the discussion last lecture, we expect 2 different solutions to this problem. First order linear, constant coefficient, homogeneous problems had exponential solutions, so let's look for exponential solutions here also:

$$y = e^{rt}$$

Homogeneous Linear Constant Coefficient Equations-II

Auxilliary Equation

The auxilliary (characteristic) equation

$$Ar^2 + Br + C = 0$$

with solutions

$$r = r_{1,2} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

Three cases:

- Two distinct real roots if $B^2 - 4AC > 0$.
- A single (repeated) real root if $B^2 - 4AC = 0$.
- Distinct complex conjugate roots if $B^2 - 4AC < 0$.

Real Roots

Case of Two Distinct Real Roots

$$A\frac{d^2y}{dt^2} + B\frac{dy}{dt} + Cy = 0$$

Auxilliary Equation

$$Ar^2 + Br + C = 0$$

has two real roots r_1 and r_2. Thus, $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$ are (different) solutions so the general solution is

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Real Roots-II

Example 1

Find the general solution of

$$y'' - 2y' - 2y = 0$$

Real Roots-III

Example 2

Solve the IVP

$$y'' + 3y' + 2y = 0 \text{ with } y(0) = 1,\ y'(0) = -1$$

Real Roots-IV

Real Repeated Root

$$r = r_{1,2} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

In the case of a repeated root, the root is $r = -B/(2A)$ and $B^2 - 4AC = 0$. One homogeneous solution is

$$y_1(t) = e^{rt}$$

We need another solution. It can be shown that in this case,

$$y_2(t) = te^{rt}.$$

Real Roots-V

Real Repeated Root (cont.)

Real Roots-VI

Example 3

Solve the IVP

$$y'' - 6y' + 9y = 0 \text{ with } y(0) = -1,\ y'(0) = 1$$

Real Roots-VI

Large Time Qualitative Behaviour

In general we could have any combination of real exponents:

- If $0 < r_1 \leq r_2$ then the solution grows exponentially and at large times the term $c_2 e^{r_2 t}$ (or $c_2 t e^{rt}$ for repeated root) will dominate at large times.
- If $r_1 \leq r_2 < 0$ then the solution decays exponentially to zero. The first term decays faster and so the solution will be dominated by the term $c_2 e^{r_2 t}$ (or $c_2 t e^{rt}$ for repeated root)
- If $r_1 < 0 < r_2$ then in general the solution grows exponentially and will be dominated by $c_2 e^{r_2 t}$ at large times. However, for initial conditions that give $c_2 = 0$ exactly, the solution will decay exponentially. However, in applications, noise will always generate the exponentially growing behaviour.

Complex Roots

$$A\frac{d^2y}{dt^2} + B\frac{dy}{dt} + Cy = 0$$

Auxilliary Equation

$$Ar^2 + Br + C = 0$$

Complex roots occur when $B^2 - 4AC < 0$, then

$$\begin{aligned} r_1 &= a + ib \\ r_2 &= a - ib \end{aligned}$$

where

$$\begin{aligned} a &= -\frac{B}{2A} \\ b &= \frac{\sqrt{4AC - B^2}}{2A} \end{aligned}$$

Complex Roots-II

Complex Exponentials

Our solutions are

$$\begin{aligned} y_1(t) &= e^{(a+ib)t} \\ y_2(t) &= e^{(a-ib)t} \end{aligned}$$

We need to make sense of the complex exponential to proceed:

$$e^{a+ib} = e^a(\cos b + i \sin b)$$

(definition). This satisfies all the properties of the "real" exponential

- $e^{a_1+b_1 i} e^{a_2+b_2 i} = e^{(a_1+a_2)+(b_1+b_2)i}$
- This justifies our use of the complex exponential solution:

$$\frac{d}{dt} e^{(a+ib)t} = (a+ib)e^{(a+ib)t}$$

Complex Roots-III

Solution

Complex solutions:

$$\begin{aligned} y_1(t) &= e^{(a+ib)t} = e^{at}(\cos bt + i \sin bt) \\ y_2(t) &= e^{(a-ib)t} = e^{at}(\cos bt - i \sin bt) \end{aligned}$$

are not nice to use to solve “real” problems. We can take linear combinations of these:

$$\begin{aligned} \frac{y_1(t) + y_2(t)}{2} &= e^{at} \cos bt \\ \frac{y_1(t) - y_2(t)}{2i} &= e^{at} \sin bt \end{aligned}$$

(like a change of basis). Thus, the general solution of the complex root case can be written

$$y(t) = c_1 e^{at} \cos bt + c_2 e^{at} \sin bt$$

Complex Roots-IV

Example 1

Solve the IVP

$$y'' + 2y' + 5y = 0 \;\text{ with } y(0) = 1,\; y'(0) = -1$$

Complex Roots-V

Example 2

Solve the IVP

$$y'' + 9y = 0 \;\text{ with } y(0) = 0,\; y'(0) = 3$$

Complex Roots-VI

Large Time Qualitative Behaviour

$$r_{1,2} = a \pm ib$$

$$y(t) = c_1 e^{at} \cos bt + c_2 e^{at} \sin bt$$

- $a < 0$ exponentially decaying periodic oscillation.
- $a = 0$ sustained periodic oscillation.
- $a > 0$ exponentially growing periodic oscillation.

Outline

Inhomogeneous Problems

Consider linear, second order constant coefficient problems with nonzero right hand sides (external forcing):

$$A\frac{d^2y}{dt^2} + B\frac{dy}{dt} + Cy = g(t)$$

where $g(t)$ is given. Remember that the general solution $y(t)$ can be written as

$$y(t) = y_p(t) + y_c(t)$$

where $y_p(t)$ is any particular solution of the equation and $y_c(t)$ is the general solution of the homogeneous (complementary) equation, which we learned how to find in the last section.

Inhomogeneous Problems-II

Method of Undetermined Coefficients

$$A\frac{d^2y}{dt^2} + B\frac{dy}{dt} + Cy = g(t)$$

If $g(t)$ has one of the following forms, the Method of Undetermined Coefficients can be used to find the particular solution:

$g(t)$ is a polynomial in t of order n: take $y_p(t)$ to also be a polynomial in t of order n.

$g(t) = \sin\omega t$ or $g(t) = \cos\omega t$: take

$$y_p(t) = a\sin\omega t + b\cos\omega t.$$

$g(t) = e^{bt}$: take

$$y_p(t) = ae^{bt}.$$

Inhomogeneous Problems-III

Method of Undetermined Coefficients (cont.)

combinations: If the RHS is an additive or multiplicative combination of the forms above, take y_p to be the additive or multiplicative combination of the correpsonding trial functions above. Additive combinations can be solved for separately.

special case (resonance): If any one of the terms in the form for the particular solution above is in the homogeneous solution, multiply the form of $y_p(t)$ above by t until this is no longer true.

solving for the coefficients: Insert the form of $y_p(t)$ into the differential equation and match functions of t to get a linear system for the undetermined coefficients in $y_p(t)$. After the coefficients are determined, then (and only then) find the complete solution $y = y_o + y_p$ using the initial data.

Examples

Example 1

Find a particular solution of

$$y'' - 3y' - 4y = 2\sin t$$

Examples-II

Example 2

Find a particular solution of

$$y'' - 3y' - 4y = te^{2t}$$

Examples-III

Example 3

Find a particular solution of

$$y'' - 3y' - 4y = te^{2t} + 2\sin t$$

Examples-IV

Example 4

Find the general solution of

$$y'' + 5y' + 4y = e^{-4t}$$

Examples-V

Example 5

Find the solution of the IVP

$$y'' + 4y' + 4y = e^{-2t} \;\text{ with } y(0) = 0 \text{ and } y'(0) = 0$$

Examples-VI

Example 5 (cont.)

Examples-VII

Example 6

Find the solution of the IVP

$$y'' + 4y = \sin \omega t \quad \text{with } y(0) = 1 \text{ and } y'(0) = 0$$

as a function of ω and t. For what values of ω does the IVP have solutions that become unbounded as $t \to \infty$?

Examples-VIII

Example 6 (cont.)

Outline

Method of Reduction of Order

Set-up

We are considering now non-homogeneous problems for linear equations that are not necessarily constant coefficient:

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = g(t)$$

where $p(t)$, $q(t)$, and $g(t)$ are given functions.

- Suppose you know one (nozero) solution $y_1(t)$ of the *homogeneous* problem (given or guessed).
- The Method of Reduction of Order will find the general solution of the non-homogeneous problem.
- By setting $g \equiv 0$ you can use the method to find a second homogeneous solution.

Method of Reduction of Order-II

Procedure

The method results in a formula, but not one that you will want to memorize. You should learn the procedure we'll work through below. The method starts by looking for a solution to the problem of the form

$$y(t) = u(t)y_1(t)$$

where u is to be determined. It is called the Method of Reduction of Order because it will be shown that u' solves a linear first order equation.

Method of Reduction of Order-III

Procedure (cont.)

Method of Reduction of Order-IV

Procedure (cont.)

Method of Reduction of Order-V

Procedure (cont.)

Examples

Example 1

$$y'' - 6y' + 9y = 0$$

has a homogeneous solution $y_1(t) = e^{3t}$. Use the Method of Reduction of Order to find a second homogeneous solution.

Examples-II

Example 2

Find the general solution of

$$y'' + y = \sin t$$

given that $y_1(t) = \sin t$ is a solution of the homogeneous problem.

Examples-III

Example 2 (cont.)

Examples-IV

Example 3

Find the general solution of

$$t^2 y'' - 2ty' + 2y = 4t^2$$

given that $y_1(t) = t$ is a solution of the homogeneous problem.

Examples-V

Example 3 (cont.)

Examples-VI

Example 4

Consider the equation

$$t^2y'' - 2y = 3t^2 - 1$$

1. Show that the homogeneous equation has a solution of the form $y = t^n$ (n to be determined).
2. Find the general solution.

Examples-VII

Example 4 (cont.)

Examples-VIII

Example 4 (cont.)