Mech 221: Computer Pre-Lab 6

Hand in the solutions to the two questions in the pre-lab at the *beginning* of the lab.

In the upcoming lab, we will work with higher order linear systems of differential equations. These arise in mechanical systems (coupled spring-mass systems) and electrical circuits (coupled RLC circuits). We will see how eigen-analysis can be useful to understand these systems. The pre-lab will cover the following:

- Introduction to the MATLAB command `eig` that computes the eigenvalues and eigenvectors of a matrix.
- Review of the amplitude-phase form of solutions of damped spring-mass systems.
- Application of eigen-analysis to the matrix from the equation of a damped spring-mass system written as a first order system. The results can be used to understand the behaviour of the system.
- Continuing to build your skill at rewriting second order differential equations as systems of first order differential equations. Here, you will be rewriting two coupled second order equations that describe a mass-spring system with two degrees of freedom as a first order system with four unknowns.

The MATLAB command `eig`

When applied to a square matrix $\mathbf{A}$, `eig(A)` will return the eigenvalues and the eigenvectors of $\mathbf{A}$. To use this command enter the following in MATLAB:

```
>> [P,D] = eig(A)
```

What will be returned is a matrix `P` with the normalized (unit length) eigenvectors of `A` in its columns, and a matrix `D` with the eigenvalues of `A` along it's diagonal. The eigenvalue corresponding to the i^{th} column of `P` is found in the (i, i) position of `D`.

Example 1: Consider A

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 5 \\ 6 & 3 & 9 \\ 2 & 7 & 8 \end{bmatrix} \tag{1}$$

We can enter `A` into MATLAB and find its eigenvectors and eigenvalues with the following commands:

```
>> A=[1 4 5; 6 3 9; 2 7 8];
>> [P,D] = eig(A)

P =

   -0.3919   -0.5895    0.2238
   -0.6401   -0.5446   -0.8511
   -0.6609    0.5966    0.4750


D =

   15.9657         0         0
         0   -0.3653         0
         0         0   -3.6004
```

These results tell us that A has eigenvectors $\{v_1, v_2, v_3\}$ and corresponding eigenvalues $\{\lambda_1, \lambda_2, \lambda_3\}$ as follows:

$$\begin{aligned} \{v_1, v_2, v_3\} &= \left\{ \begin{pmatrix} -0.3919 \\ -0.6401 \\ -0.6609 \end{pmatrix}, \begin{pmatrix} -0.5895 \\ -0.5446 \\ 0.5966 \end{pmatrix}, \begin{pmatrix} 0.2238 \\ -0.8511 \\ 0.4750 \end{pmatrix} \right\} && (2) \\ \{\lambda_1, \lambda_2, \lambda_3\} &= \{15.9657, -0.3653, -3.6004\} && (3) \end{aligned}$$

Review and Eigen-Analysis of a Damped Spring-Mass System

Consider a simple harmonic oscillator with damping. The governing equation is given by:

$$m\ddot{x} + \beta\dot{x} + kx = 0, \quad m, b, k > 0 \tag{4}$$

where x is the displacement of the oscillator with mass m from equilibrium. β and k are the damping and spring constants respectively. Let us consider a specific example:

Example 2: Consider a scaled version of (4) that leads to the equation

$$\ddot{x} + \gamma\dot{x} + x = 0 \tag{5}$$

where $\gamma = 0.1$.

To find solutions of (5) in Example 2 above we write the auxiliary equation

$$r^2 + 0.1r + 1 = 0$$

which has complex roots $r_{1,2} = a \pm ib$ with $a = -0.05$ and $b \approx 0.9987$. The fact that the roots are complex means that the system described by (5) is under-damped. We can now write the general solution of (5):

$$x(t) = e^{at}\left(c_1 \cos(bt) + c_2 \sin(bt)\right)$$

or written in amplitude-phase form

$$x(t) = Re^{at} \cos(bt + \phi). \tag{6}$$

Here, the two constants R and ϕ replace the two constants c_1 and c_2 in the first form in the description of the solution. The angular frequency b and the decay rate a of solutions to the damped spring-mass system are of physical importance. For this single spring-mass system, these are easily found as roots of the quadratic auxiliary equation as done above. They can be estimated from solution curves as shown in question #1 below. An alternate technique to find the angular frequency b and the decay rate a of solutions using eigen-analysis is described below. This technique can be applied to spring-mass systems with more than one degree of freedom as shown later in the pre-lab.

If we wanted to compute numerical solutions of (5) of Example 2 we would write it as a first order system in the usual way:

$$\frac{d}{dt}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} \dot{x} \\ -x - 0.1\dot{x} \end{bmatrix}. \tag{7}$$

Note that the right hand side of the vector equation above can be written as matrix multiplication

$$\mathbf{A}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} \quad \text{where} \quad \mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & -0.1 \end{bmatrix}.$$

It is always possible to write first order, linear, constant coefficient, inhomogeneous differential equations in terms of matrix multiplication. Such systems can be solved using eigen-analysis. If the matrix $\mathbf{A}$ is typed into

MATLAB and its eigenvalues calculated, the results are $-0.0500 \pm i0.9987$. Note that these are exactly $a \pm ib$ for the problem written as a second order scalar equation. Convince yourself that this is what should happen.

Question 1: Identifying Frequency and Decay Rate from Solution Plots

The figure below contains four plots of functions of the form (6). Match the plots to the values of b and a below.

(a) $a = 0$, $b = 1$

(b) $a = -0.1$, $b = 1$

(c) $a = -0.1$, $b = 2$

(d) $a = -1$, $b = 2$

- *Hand in the match of plots to values of a and b listed above as your answer for this question.*

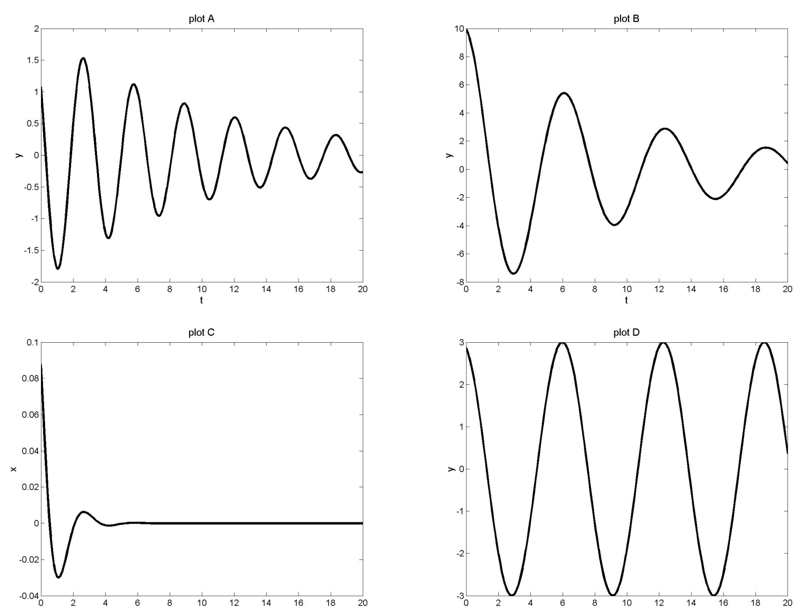

Figure 1: Plots to match to values of a and b in Question #1

Coupled Oscillators

Consider now two masses m_1, m_2 in a box. Let them be mutually coupled by a spring with spring constant K. Further, let m_1 be coupled to a wall by a spring of constant k_1 and let m_2 be coupled to the opposite wall by a spring of constant k_2, so that both masses are in a line as shown in Figure 2. Let $x(t)$ and $y(t)$ be the displacements of masses m_1, m_2 from their corresponding equilibrium positions in this configuration.

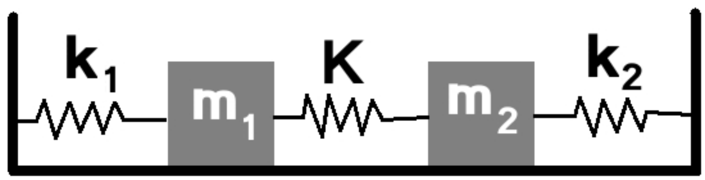

Figure 2: Coupled spring mass system with two masses.

Using Newton's Second Law, we can derive the following pair of differential equations that govern their behaviour.

$$\begin{aligned} m_1\ddot{x} &= -k_1 x - K(x-y) - \beta_1 \dot{x} \\ m_2\ddot{y} &= K(x-y) - k_2 y - \beta_2 \dot{y} \end{aligned} \tag{8}$$

where β_1 and β_2 are the coefficients of damping in the motion of the corresponding masses. Note that these damping terms are not the same as the frictional forces in the earlier lab. If the (scaled) values of β_1 and β_2 are small, the solution component $x(t)$ above will have the form

$$x(t) = R_1 e^{a_1 t}\cos(b_1 t + \phi_1) + R_2 e^{a_2 t}\cos(b_2 t + \phi_2). \tag{9}$$

Note that this solution is a superposition of two decaying, oscillatory parts with (in general) different decay rates $a_{1,2}$ and angular frequencies $b_{1,2}$. The solution for $y(t)$ has a similar form. The values $a_{1,2}$ and $b_{1,2}$ are important in an application (the decay rates and angular frequencies of oscillations in the system). An example of such a solution is shown in Figure 3. You can see that it would be hard to identify the values of $a_{1,2}$ and $b_{1,2}$ from such a plot as we were able to do in Question #1. You will learn how to do this with eigen-analysis in Lab #6.

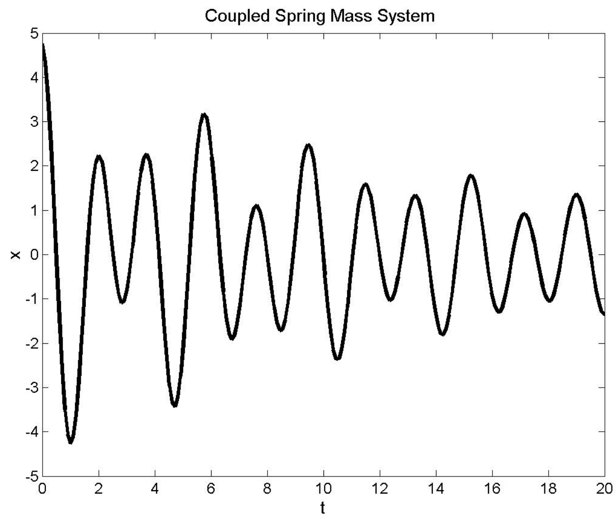

Figure 3: An example solution to the coupled oscillator (8) in the form (9).

Question 2: Coupled Oscillators

- Write the second order system (8) as a first order vector valued differential equation. (Hint: Let $x_1 = x, x_2 = \dot{x}, x_3 = y, x_4 = \dot{y}$)
- The system you wrote above can be written in matrix form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

  where $\mathbf{x}$ is the column vector of the four components x_1, x_2, x_3 and x_4.
- *Hand in the matrix* $\mathbf{A}$ *as your answer to this question*

Note: For specific values of m_1, m_2, k_1, k_2, K, β_1 and β_2 the 4×4 matrix $\mathbf{A}$ can be entered into MATLAB and its eigenvalues computed. If the damping is small, the eigenvalues of $\mathbf{A}$ will come in two complex pairs $a_1 \pm ib_1$ and $a_2 \pm ib_2$. Thus, eigen-analysis of $\mathbf{A}$ will give us the decay rates and angular frequencies we are interested in.

Additional Note: The matrix of eigenvectors of $\mathbf{A}$ can be used to solve (8) with initial conditions. It also gives a description how the oscillations for x and y are coupled. Due to time constraints on this lab, we won't be working with the eigenvectors (although it would be interesting and useful).

Coming up in Lab #6

Completing the pre-lab will prepare you for the lab, in which the following will be covered:

- Using eigenvalues to assess qualitative behaviour of coupled systems of oscillators.